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## Equivalence between first-order causal and stable hydrodynamics and Israel-Stewart theory for boost-invariant systems with a constant relaxation time

Arpan Das<sup>a,\*</sup>, Wojciech Florkowski<sup>b</sup>, Jorge Noronha<sup>c</sup>, Radoslaw Ryblewski<sup>a</sup><sup>a</sup> Institute of Nuclear Physics Polish Academy of Sciences, PL-31-342 Krakow, Poland<sup>b</sup> Institute of Theoretical Physics, Jagiellonian University, PL-30-348 Kraków, Poland<sup>c</sup> Department of Physics, University of Illinois at Urbana-Champaign, IL 61801-3080, USA

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## ABSTRACT

We show that the recently formulated causal and stable first-order hydrodynamics has the same dynamics as Israel-Stewart theory for boost-invariant, Bjorken expanding systems with an ideal gas equation of state and a regulating sector determined by a constant relaxation time. In this case, the general solution of the new first-order formulation can be determined analytically.

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## 1. Introduction

Relativistic hydrodynamics has become nowadays the basic theoretical tool for modeling relativistic heavy-ion collisions [1,2]. It forms the main ingredient of the so-called standard model of such processes, which essentially includes three segments: modeling of the early stage, hydrodynamic description of the space-time evolution of matter, and freeze-out of hadrons [3–6]. Detailed comparisons of theoretical predictions based on the hydrodynamic approach with the experimental data allow for the determination of various properties of strongly interacting matter such as its equation of state [7] and kinetic coefficients [8–10]. The latter include the shear and bulk viscosities. The presence of the shear viscosity affects the response of the hydrodynamic flow to the initial space-time anisotropies of colliding matter [11,12].

The development of hydrodynamic models for the description of heavy-ion collisions triggered broad studies of formal aspects of hydrodynamics treated as an effective theory describing systems approaching local thermodynamic equilibrium, for a recent review see [13]. Already in the 1970's, it was realized that the relativistic dissipative hydrodynamical formulations derived by Landau

and Eckart were not causal [14–16] and they were replaced by the so-called second order hydrodynamic formalism of Israel and Stewart (IS) [17]. The IS theory has been extensively used to describe heavy-ion collisions studied at the Relativistic Heavy Ion Collider (RHIC) at BNL and the Large Hadron Collider (LHC) at CERN. At the same time, more advanced hydrodynamic approaches have been developed, which removed some of the disadvantages of the IS formulation (for example, see [18–22]). Formal studies of hydrodynamics have led to very interesting observations such as the asymptotic character of the hydrodynamic gradient expansion [23–26] or the existence of hydrodynamic attractors [27–32].

IS theory treats the shear stress tensor  $\pi^{\mu\nu}$  and the bulk pressure  $\Pi$  as independent hydrodynamic variables, in a way similar to the treatment of the local temperature  $T(x)$  and the hydrodynamic flow vector  $u^\mu(x)$ . Only during the space-time evolution of the system  $\pi^{\mu\nu}$  and  $\Pi$  may approach their Navier-Stokes values  $\pi^{\mu\nu} = 2\eta\sigma^{\mu\nu}$  and  $\Pi = -\zeta\partial_\mu u^\mu$  (where  $\eta$  and  $\zeta$  are the shear and bulk viscosity coefficients, respectively, and  $\sigma^{\mu\nu}$  is the shear flow tensor constructed from the derivatives of  $u^\mu$ ).

Only very recently, a new causal and stable hydrodynamic approach based on a first-order expansion in derivatives has been proposed by F. S. Bemfica, M. M. Disconzi, J. Noronha, and P. Kovtun [33–35]. This approach is based on a more general choice of the hydrodynamic frame and the introduction of a new set of kinetic coefficients that play the role of UV regulators of the theory, which make the theory causal (even in the full nonlinear regime) and linearly stable around equilibrium.

\* Corresponding author.

E-mail addresses: [arpan.das@ifj.edu.pl](mailto:arpan.das@ifj.edu.pl) (A. Das), [wojciech.florkowski@uj.edu.pl](mailto:wojciech.florkowski@uj.edu.pl) (W. Florkowski), [jn0508@illinois.edu](mailto:jn0508@illinois.edu) (J. Noronha), [radoslaw.ryblewski@ifj.edu.pl](mailto:radoslaw.ryblewski@ifj.edu.pl) (R. Ryblewski).

A natural question that can be asked is how the new formulation (dubbed below shortly as FOCS, for first-order causal and stable) compares to the traditional IS framework. It was shown in Ref. [33] that the two approaches lead to very similar equations, if applied to boost-invariant conformal systems. In this work we extend this study. We assume that the system's equation of state is that of an ideal gas but we allow for a non-conformal behavior of the coefficients in the regulating sector of the theory. We show that if the kinetic coefficients are expressed in terms of a constant relaxation time there is an exact match between the dynamics described by FOCS and IS formulations. This allows us to derive the first general analytical solution of the FOCS equations for an expanding system.

Throughout the paper we use natural units.

## 2. Israel-Stewart and first-order causal and stable hydrodynamics

The implementation of Israel-Stewart theory undergoing Bjorken flow [36] considered here is reduced to the two equations:

$$\frac{d\varepsilon}{d\tau} = -\frac{\varepsilon + p}{\tau} + \frac{\pi}{\tau}, \quad (1)$$

$$\tau_R \frac{d\pi}{d\tau} + \pi = \frac{4}{3} \frac{\eta}{\tau} - \left( \frac{4}{3} + \lambda \right) \tau_R \frac{\pi}{\tau}, \quad (2)$$

where  $\varepsilon$  and  $p$  are the energy density and pressure,  $\pi$  is the rapidity-rapidity component of the shear stress tensor (which should not be mistaken with the bulk pressure,  $\Pi$ , that is zero in our case),  $\eta$  is the shear viscosity coefficient,  $\tau_R$  is the relaxation time, and the parameter  $\lambda$  [37] is related to the  $\tau_{\pi\pi}$  coefficient in the DNMR approach [38]. The evolution parameter  $\tau = \sqrt{t^2 - z^2}$  is the longitudinal proper time. We note that the form of hydrodynamic flow for boost-invariant systems is dictated by symmetry,  $u^\mu = (t/\tau, 0, 0, z/\tau)$ , hence it is independent of the choice of the hydrodynamic frame. We also note that all scalar functions depend only on  $\tau$ ,  $\partial_\mu u^\mu = 1/\tau$ , and  $u^\mu \partial_\mu f(\tau) = df/d\tau$ , where  $f$  is an arbitrary function of the proper time  $\tau$ .

Throughout this paper we assume the ideal gas equation of state

$$p = \frac{1}{3} \varepsilon = \frac{aT^4}{3}, \quad (3)$$

where  $a$  is a constant (usually proportional to the number of internal degrees of freedom of the particles forming a fluid) and  $T$  is the temperature. We can thus rewrite Eq. (1) as

$$\frac{dT}{d\tau} = \frac{\pi}{4aT^3\tau} - \frac{T}{3\tau}. \quad (4)$$

On the other hand, introducing the variable

$$y = \frac{dT}{d\tau} \quad (5)$$

and taking the derivative of Eq. (4) with respect to  $\tau$ , we obtain

$$\frac{d\pi}{d\tau} = 4aT^3 y + 4aT^3 \tau \frac{dy}{d\tau} + 12aT^2 \tau y^2 + \frac{16}{3} aT^3 y. \quad (6)$$

This allows us to rewrite Eq. (2) as

$$\begin{aligned} & 4a\tau_R T^3 \frac{dy}{d\tau} + 12\tau_R aT^2 y^2 \\ & + aT^3 y \left[ 4 + \left( \frac{28}{3} + 4 \left( \frac{4}{3} + \lambda \right) \right) \frac{\tau_R}{\tau} \right] \\ & + \frac{4aT^4}{3\tau} + \frac{4}{3} aT^4 \left( \frac{4}{3} + \lambda \right) \frac{\tau_R}{\tau^2} - \frac{4}{3} \frac{\eta}{\tau^2} = 0. \end{aligned} \quad (7)$$

Equations (5) and (7) are coupled differential equations for the functions  $T$  and  $y$ , which are completely equivalent to the original IS equations. We note that Eq. (7) has the form of a Riccati equation ( $ay' + by^2 + cy + d = 0$ , with  $b/a \neq 0$  and  $c/a \neq 0$ ), which was analyzed recently in more detail in [37].

For the FOCS approach [33–35], the evolution equations in Bjorken flow are reduced to the formula

$$\frac{d\mathcal{E}}{d\tau} + \frac{\mathcal{E} + \mathcal{P}}{\tau} - \frac{4}{3} \frac{\eta}{\tau^2} = 0, \quad (8)$$

where the following constitutive relations are assumed,

$$\begin{aligned} \mathcal{E} &= \varepsilon + \varepsilon_1 \frac{dT}{T d\tau} + \frac{\varepsilon_2}{\tau}, \\ \mathcal{P} &= p + \pi_1 \frac{dT}{T d\tau} + \frac{\pi_2}{\tau}, \end{aligned} \quad (9)$$

where again  $\varepsilon = 3p$ . In natural units, the regulating coefficients in FOCS  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\pi_1$ , and  $\pi_2$  have dimension of energy cubed.

## 3. Regulating sector in FOCS

Let us discuss in more detail the regulating sector of the FOCS approach. We first remind the reader that in conformal fluids in 4 spacetime dimensions the energy-momentum tensor  $T^{\mu\nu}$  of the system must be traceless and it must also change homogeneously under Weyl transformations of the metric, i.e.,  $T^{\mu\nu} \rightarrow e^{6\omega} T^{\mu\nu}$  when  $g_{\mu\nu} \rightarrow e^{-2\omega} g_{\mu\nu}$  [39]. This implies that in a conformal fluid described by the FOCS approach all the coefficients should scale as  $T^3$  with  $\varepsilon_1 = 3\varepsilon_2$  and  $\pi_1 = 3\pi_2$  to ensure invariance under Weyl transformations [33], together with  $\mathcal{E} = 3\mathcal{P}$ , which in turn sets  $\pi_i = \varepsilon_i/3$  ( $i = 1, 2$ ). On the other hand, for conformal IS theory the IS relaxation time  $\tau_R$  should be inversely proportional to  $T$ , while  $\eta$  again should scale with  $T^3$ , yielding a dimensionless ratio of the shear viscosity  $\eta$  to the entropy density  $s$ .

In this work we want to discuss yet another case, where the coefficients  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\pi_1$ , and  $\pi_2$  are expressed in terms of a constant relaxation time. This leads to parametrizations of the type  $x_i = x_i^0 T^4$ , where  $x_i$  stands for any of the FOCS coefficients mentioned above and  $x_i^0$  has dimension of time (fm). We think that this assumption is interesting from the point of view where the terms containing  $\varepsilon_1$ ,  $\varepsilon_2$ ,  $\pi_1$ , and  $\pi_2$  are interpreted as ultraviolet regulators. In general, the regularization or renormalization procedure of a classically scale-invariant theory introduces an energy scale, as it happens in the case of pure Yang-Mills theory (known exceptions of this rule include, for instance,  $\mathcal{N} = 4$  supersymmetric Yang-Mills theory).

To include and discuss different cases together we rewrite Eq. (9) as

$$\begin{aligned} \mathcal{E} &= aT^4 + \varepsilon_1^0 T^n \frac{dT}{T d\tau} + \frac{\varepsilon_2^0}{\tau} T^n, \\ \mathcal{P} &= \frac{aT^4}{3} + \pi_1^0 T^n \frac{dT}{T d\tau} + \frac{\pi_2^0}{\tau} T^n, \end{aligned} \quad (10)$$

where  $\varepsilon_1^0$ ,  $\varepsilon_2^0$ ,  $\pi_1^0$ , and  $\pi_2^0$  can be dimensionless for  $n = 3$  or dimensionful quantities for  $n \neq 3$ . The power  $n$  can take different values depending on the case we want to discuss. Substituting (10) into Eq. (8), and using Eq. (5), we find

$$\begin{aligned} & \varepsilon_1^0 T^{n-1} \frac{dy}{d\tau} + (n-1) \varepsilon_1^0 T^{n-2} y^2 \\ & + \left( 4aT^3 + (\varepsilon_1^0 + \pi_1^0 + n\varepsilon_2^0) \frac{T^{n-1}}{\tau} \right) y \\ & + \frac{4}{3} aT^4 + \frac{\pi_2 T^n}{\tau^2} - \frac{4}{3} \frac{\eta}{\tau^2} = 0. \end{aligned} \quad (11)$$

Equations (5) and (11) are coupled first-order differential equations that can be treated as the basis of the FOCS formulation in our setup.

#### 4. Comparison between the two frameworks

Using the parametrizations defined above we can formulate the IS and FOCS frameworks in terms of the two differential equations for the temperature  $T$  and its derivative  $y = dT/d\tau$ . A natural question is if these two formulations, when written in this form, are actually identical describing thus the same dynamics. Since Eq. (5) is common for the two approaches, one simply has to check if Eqs. (7) and (11) are equivalent. After equating the terms with the same derivatives of the function  $y$  in Eqs. (7) and (11) we find:

$$\varepsilon_1^0 = 4a\tau_R T^{4-n}, \quad (12)$$

$$\varepsilon_1^0 = \frac{12}{n-1} a\tau_R T^{4-n}, \quad (13)$$

$$\pi_1^0 = \frac{4}{3} a\tau_R (11 + 3\lambda) T^{4-n} - \varepsilon_1^0 - n\varepsilon_2^0, \quad (14)$$

$$\pi_2^0 = \frac{4}{9} a\tau_R (4 + 3\lambda) T^{4-n}. \quad (15)$$

One can easily notice that in the conformal case,  $n = 3$ , it is impossible to exactly match the FOCS and IS equations, even though the evolution equation for  $y$  in both formulations can be written as a Riccati equation. The parametrization of the Riccati equation found in the conformal case in [33] uses a function  $\chi$  that is related to our parametrization through the formula

$$\chi = \frac{1}{3} \varepsilon_1^0 T^3 = \varepsilon_2^0 T^3. \quad (16)$$

Moreover, in [33] one uses the relation  $\pi_1^0 = (1/3)\varepsilon_1^0 = \varepsilon_2^0$ .

A very interesting situation takes place when  $n = 4$ . In this case Eqs. (12) and (13) are fully consistent and the kinetic coefficient  $\varepsilon_1^0$  has dimension of fm and, thus, it can be treated as a fixed relaxation time related to  $\tau_R$  (which is also constant). Equations (14) and (15) determine the values of  $\pi_1^0$  and  $\pi_2^0$  in terms of the IS relaxation time,  $\varepsilon_1^0$  and  $\varepsilon_2^0$ . Although for  $n = 4$  the system of equations (12)–(15) can be adjusted to exactly match the IS equations, at first it seems that the matching is underdetermined as only the sum  $\pi_1^0 + 4\varepsilon_2^0$  is constrained by Eq. (14). We discuss this apparent issue in more detail below.

##### 4.1. Tracelessness constraint

Further insights about the identification of the FOCS and IS approaches can be gained from the fact that the energy-momentum tensor of the IS theory we consider is traceless. We stress that this is not inconsistent with having a constant relaxation time  $\tau_R$ . As mentioned above, a fluid is conformal if its  $T^{\mu\nu}$  is traceless and if  $T^{\mu\nu}$  transforms homogeneously under Weyl transformations. The latter property does not hold in the system when  $\tau_R$  is constant so conformal invariance is broken despite the use of the ideal gas equation of state.

In order to better define the mapping between the IS and FOCS approaches, it is reasonable to impose that the energy-momentum tensor in the FOCS approach is also traceless. This implies that we should consider only the case where  $\pi_i = \varepsilon_i/3$  ( $i = 1, 2$ ). As we show below, this guarantees that the bulk viscosity is zero in this approach.

Imposing then that  $\pi_i = \varepsilon_i/3$  ( $i = 1, 2$ ) in (12)–(15), with  $n = 4$ , gives that the IS parameter  $\lambda = -1$  and

$$\varepsilon_1^0 = 4a\tau_R, \quad (17)$$

$$\varepsilon_2^0 = \frac{4}{3} a\tau_R, \quad (18)$$

$$\pi_1^0 = \frac{4}{3} a\tau_R, \quad (19)$$

$$\pi_2^0 = \frac{4}{9} a\tau_R. \quad (20)$$

Using the notation for the FOCS coefficients in [34], we see that

$$3\chi_1 = \varepsilon_1^0 T^4, \quad \chi_2 = \varepsilon_2^0 T^4, \quad 3\chi_3 = \pi_1^0 T^4, \quad \chi_4 = \pi_2^0 T^4, \quad (21)$$

and it is easy to check that the bulk viscosity coefficient in FOCS [35,34]

$$\zeta = \chi_3 - \chi_4 + c_s^2 (\chi_2 - \chi_1) \quad (22)$$

vanishes when one plugs in the values in (17)–(20), as expected (note that  $c_s^2 = 1/3$  for the ideal gas equation of state). Therefore, one can see that the tracelessness condition removes the apparent ambiguity in the mapping present in Eq. (14).

##### 4.2. Constraints from causality and linear stability in the FOCS formulation

For the type of Israel-Stewart theory considered here, causality and stability around equilibrium hold if  $\eta/(s\tau_R T) \leq 1/2$  (where  $s = 4\epsilon/3T$ ) [40]. We note that this is a statement obtained after linearizing the equations around equilibrium and, thus, no constraint is known for the  $\lambda$  coefficient, as it does not contribute in a linearized analysis. However, this coefficient is known in the 14-moment approximation to be equal to  $10/21$  [38], while the shear viscosity is given by  $\eta = 4\varepsilon\tau_R/15$ .

Ref. [34] derived conditions for the transport coefficients in the FOCS approach that ensure causality in the full nonlinear regime. Also, linear stability conditions around equilibrium were discussed in both [34] and [35]. We refer the reader to Section III A and B of Ref. [34] for the set of inequalities that must be fulfilled for causality and stability to hold in the FOCS approach. Also, we note that in the FOCS formalism no further conditions appear from the second law of thermodynamics besides the usual statement that  $\eta, \zeta \geq 0$  [35,34].

Since the causality conditions in the full nonlinear regime are known for the first-order approach, it is interesting to consider if the identifications made in (17)–(20) can fulfill the conditions stated in [34]. Using the 14 moments relation between  $\eta$  and  $\tau_R$  we see that the non-equilibrium energy correction coefficient in FOCS can be written here as  $\chi_1 = 5\eta$ . One can show that causality and stability in the FOCS theory hold for the coefficients given by (17)–(20) when the energy flow coefficient in [34] equals  $15\eta/4$ .

We remark that it is not currently known if the 14-moment value for  $\lambda$  in IS theory leads to causality violations once the full nonlinear dynamics of the equations is taken into account. The results presented herein may suggest that the IS parameter  $\lambda = 10/21$  can be at odds with causality when one goes beyond the linearized regime. However, such a conjecture can only be checked once a full nonlinear analysis of causality in Israel-Stewart theory, performed under general conditions, is available. So far, such general statements about causality in the nonlinear regime of Israel-Stewart theory have been obtained in [41] in the case where only bulk viscosity (i.e., no shear or particle diffusion effects) is taken into account (a nonlinear study involving shear and bulk viscosities in IS theory under strong symmetry conditions can be found in [42]).

#### 4.3. General analytical solution for the FOCS approach matched to IS theory

In Ref. [37] the general solution of the IS equations (1) and (2) for the Bjorken flow have been found. The analytical expressions for  $\varepsilon(\tau)$  and  $\pi(\tau)$  can be obtained from Eqs. (15) and (16) of [37]. The matching to IS theory worked out in this paper (see (17)–(20)) implies that the general solution for the energy density in IS found in Eq. (15) of [37] also holds for the FOCS theory. Therefore, under these conditions, the general solution for the energy density in the FOCS approach, with a constant relaxation time as defined here, is

$$\varepsilon(\hat{\tau}) = \varepsilon_0 \left( \frac{\hat{\tau}_0}{\hat{\tau}} \right)^{\frac{4}{3} + \frac{\lambda+1}{2}} \exp \left( -\frac{\hat{\tau} - \hat{\tau}_0}{2} \right) \times \left[ \frac{M_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4\kappa}}{2}}(\hat{\tau}) + \alpha W_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4\kappa}}{2}}(\hat{\tau})}{M_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4\kappa}}{2}}(\hat{\tau}_0) + \alpha W_{-\frac{\lambda+1}{2}, \frac{\sqrt{\lambda^2+4\kappa}}{2}}(\hat{\tau}_0)} \right] \quad (23)$$

where  $\kappa = 16\eta/(9\tau_R Ts)$ ,  $\lambda = -1$ ,  $\hat{\tau} = \tau/\tau_R$ ,  $\hat{\tau}_0$  is the initial time,  $\varepsilon_0$  and  $\alpha$  are constants that define the initial value problem, and  $M_{k,\mu}(x)$  and  $W_{k,\mu}(x)$  are Whittaker functions. This is the first analytical solution of the viscous relativistic hydrodynamics equations derived from the new first-order approach put forward in Refs. [33–35]. It should be clear also that the mapping between these approaches found here immediately establishes the properties of the hydrodynamic attractor in the FOCS approach in this case, as they can be extracted from the analysis already performed in IS theory in [37].

#### 5. Conclusions

In this work we have compared the recent first-order causal and stable formulation of relativistic hydrodynamics with conventional Israel-Stewart theory. To make such a comparison feasible, we have restricted our study to boost-invariant, baryon-free systems with an ideal gas equation of state. In the strictly conformal case, where the regulator sectors of the theories are also determined from conformal invariance, the two approaches cannot be exactly matched, although they are based on the same system of differential equations (see Ref. [33]). If the regulator sectors of the theories are determined by a constant relaxation time, there exists a mapping between the FOCS and IS approaches that makes their dynamics exactly the same. This implies that one can use the results in [37] to determine the first general analytical solution of the FOCS equations of motion, as we showed in this paper. The causality conditions for the FOCS approach found in [34] proved to be relevant when determining the range of acceptable values of the transport coefficients in the FOCS approach, after the matching to IS theory. In fact, we showed that this matching to IS theory is only well defined if the IS parameter  $\lambda$  takes a value that is distinct from the standard 14-moment result.

Our results help to clarify mutual relations between FOCS and more traditional formulations of relativistic dissipative hydrodynamics. Further investigations of more general systems are of course mandatory in this respect. Although for more complex system simple relations connecting FOCS with second order hydrodynamic frameworks may not exist (since FOCS yields four second-order equations which are in general equivalent to eight first-order equations, while Israel-Stewart theory is based on ten equations describing the time evolution of ten independent components of the symmetric energy-momentum tensor), it is in our opinion very interesting to identify the cases where such constructions are possible. This helps to better understand the physics behind this new first-order formulation, which may eventually become an attractive alternative to more traditional hydrodynamic frameworks.

#### Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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